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On One Two-dimensional Stationary Flow of a Binary Mixture and Viscous Fluid in a Plane Layer

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Nonlinear model of convection in Oberbeck-Boussinesq approximation describing the flat joint motion of a binary mixture and viscous fluid with a common interface is investigated. It is important that the longitudinal temperature gradient and the concentration is quadratic dependence on the coordinate x. Stationary solution of the system is built.

Keywords: Oberbeck-Boussinesq equations, convective motion, binary mixture, steady flow.
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Convection is one of the most common of hydrodynamic phenomena in nature. Study of convection is an important part of the theoretical fluid mechanics. Natural convective motions occur in an inhomogeneous field of mass forces caused by the ununiform heating of the liquid. The area of practical applications of this phenomenon is very wide. Convective processes influence the thermal conditions in the oil storage tanks, chemical process technology and others. Theoretical study of natural convection usually deals with equations of motion in the Oberbeck-Boussinesq approximation. Problems for thermal convection are very complex because of the diversity of cavities and thermal boundary conditions for the nonlinear system of partial differential equations. Solutions of the Oberbeck-Boussinesq equations with a linear dependence of temperature on one of the space coordinates firstly were studied by G. A. Ostroumov [1]. The exact solution described plane stationary flow in a strip under action of longitudinal temperature gradient and transversal gravity field, was obtained by R. V. Birikh [2]. Some generalizations of this solution taking into account concentration of liquid mixture are described in [3]. The existence of solutions with nonlinear dependence of density on temperature and concentration is proved in [4] where two boundary value problems with exponential temperature distribution on the walls are solved. In [5–7] the exact solutions of the three-dimensional convection problem for two immiscible viscous, incompressible fluids in a channel with a rectangular cross section in the presence of the interface and under the influence of a longitudinal temperature gradient are studied. In this paper we are built the exact solution of the two-layer convection system with longitudinal temperature gradient at the solid walls and shear force of gravity.

1. Problem formulation

Let us consider the joint motion of a binary mixture and viscous fluid with a general interface. Suppose that $\Omega_1 = \{\, |x| < \infty,\, 0 < y < l_1\}$ is the region occupied by a binary mixture and

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$\Omega_2 = \{|x| < \infty,\, l_1 < y < l_2\}$ is the region with a viscous fluid. The system is bounded by solid walls $y = 0$ and $y = l_2$ with given temperature distribution on them.

For description of the motion of both region Ω_j $(j = 1, 2)$ we use the Boussinesq approximation. We assume that the temperature and the concentration differ only slightly from constant mean values therefore the Oberbeck-Boussinesq approximation is valid

$$\rho_j = \rho_{0j}(1 - \beta_j^\theta \theta - \beta_j^c c),$$

where ρ_{0j} is the characteristic medium density corresponding to the mean values of the temperature and concentration in the layer j, θ and c are the deviations from their mean values (c corresponds to the light component), β_j^θ and β_j^c are the temperature and concentration expansion coefficient; $\beta_2^c = 0$. Then the equations of binary mixture convection can be written in the form

$$\begin{gathered}
\mathbf{u}_{jt} + (\mathbf{u}_j \cdot \nabla)\mathbf{u}_j = -\frac{1}{\rho_{0j}}\nabla p_j + \nu_j \Delta \mathbf{u}_j - \mathbf{g}(\beta_j^\theta(\theta_j - \theta_{0j}) + \beta_j^c(c_j - c_{0j})),\\
\theta_{jt} + \mathbf{u}_j \cdot \nabla\theta_j = \chi_j \Delta\theta_j,\\
c_{1t} + \mathbf{u}_1 \cdot \nabla c_1 = d_1 \Delta c_1 + \alpha d_1 \Delta\theta_1,\\
\operatorname{div} \mathbf{u}_j = 0.
\end{gathered} \tag{1}$$

Here $\mathbf{u}_j$ is the velocity field, p_j is the pressure measured from the hydrostatic pressure corresponding to p_{0j}, ρ_{0j} is the density, ν_j is kinematic viscosity, χ_j is the temperature conductivity, d_1 is the diffusivity, αd_1 is the thermal diffusion coefficient. Normal thermal diffusion corresponds to the value of $\alpha < 0$, and for the anomalous $\alpha > 0$.

We introduce the coordinate system with the x axis aligned with the lower boundary of the layer 1 and the y axis directed vertically upward (Fig. 1).

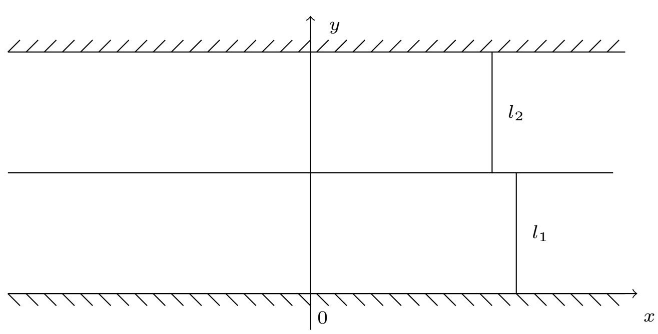

Fig. 1. The scheme of two-layer flow between the rigid walls with interface $y = l_1$

Let us define the boundary conditions. On solid walls are put no-slip conditions and the temperature distribution and the absence of mass flux through the walls are written as

$$\begin{aligned}
&y = 0: \quad \mathbf{u}_1 = 0, \quad \theta = \theta_{10}(x), \quad c_{1y} + \alpha\theta_{1y} = 0;\\
&y = l_2: \quad \mathbf{u}_2 = 0, \quad \theta = \theta_{20}(x).
\end{aligned} \tag{2}$$

At the interface $y = l_1$ the conditions of equality of the velocities, kinematic and dynamic conditions are written as

$$\begin{aligned}
&u_1 = u_2, \quad v_1 = v_2 = 0,\\
&\rho_2\nu_2 u_{2y} - \rho_1\nu_1 u_{1y} = -æ_1\theta_{1x} - æ_2 c_{1x}.
\end{aligned} \tag{3}$$

The condition of temperature continuity and the equality of heat fluxes are as follows

$$\theta_1 = \theta_2, \quad k_1\theta_{1y} = k_2\theta_{2y}. \tag{4}$$

In addition, the condition of absence of mass flux through the interface is

$$c_{1y} + \alpha\theta_{1y} = 0. \tag{5}$$

Here $k_j = \chi_j \rho_j c_{p_j}$ are the thermal conductivities, $\sigma = \sigma(\theta, c)$ is the coefficient of surface tension. For many mixtures, the linear law provides a good approximation of this dependence

$$\sigma(\theta, c) = \sigma^0 - æ_1(\theta - \theta_0) - æ_2(c_1 - c_0),$$

where $æ_1 > 0$ is the temperature coefficient and $æ_2$ is the concentration coefficient (usually $æ_2 < 0$ since the surface tension increases with concentration). Constants θ_0, c_0 are the temperature and concentration values of arbitrary point on interface.

We should to add the initial conditions: $u_j = 0$, $\theta_j = \theta_j^0(x, y)$, $c_1 = c^0(x, y)$. All the physical characteristics of the system are assumed to be constant and correspond to the mean temperature and concentration.

2. Exact solution of the two-dimensional problem

We find the form of the solution describing the convective flow in the system of liquids with the interface in the form

$$\begin{gathered} u_j = U_j(y,t)x + W_j(y,t), \quad v_j = V_j(y,t); \\ \theta_j = A_j(y,t)x^2 + B_j(y,t), \quad c_1 = H_1(y,t)x^2 + E_1(y,t), \\ p_j = P(x,y,t). \end{gathered} \tag{6}$$

The substitution of solution (6) into equations of motion (1) gives the relations

$$\begin{gathered} \nu_j U_{jyy} - U_{jt} - U_j^2 - V_j U_{jy} = 2g \int_{\Omega_j} (\beta_j^\theta A_j + \beta_j^c H_j) dy + s_j(t), \\ \frac{1}{\rho_j} P_j = (\nu_j U_{jyy} - U_{jt} - U_j^2 - V_j U_{jy}) \frac{x^2}{2} + h_j(y,t), \\ h_{jy} = \nu_j V_{jyy} - V_{jt} - V_j V_{jy} + g(\beta_j^\theta B_j + \beta_j^c E_j), \\ W_j = 0, \quad V_{jy} = -U_j. \end{gathered} \tag{7}$$

The equations for determining of the temperature and the concentration field take the form

$$\begin{gathered} A_{jt} + 2U_j A_j + V_j A_{jy} = \chi_j A_{jyy}, \\ B_{jt} + V_j B_{jy} = \chi_j \left(2A_j + B_{jyy}\right), \\ H_{1t} + 2U_1 H_1 + V_1 H_{1y} = d_1 \left(H_{1yy} + \alpha_1 A_{1yy}\right), \\ E_{1t} + V_1 E_{1y} = d_1 \left(2H_1 + E_{1yy} + \alpha_1(2A_1 + B_{1yy})\right). \end{gathered} \tag{8}$$

The following boundary condition at solid walls are held

$$\begin{gathered} U_1(0,t) = 0, \quad U_2(l_2,t) = 0, \quad A_1(0,t) = A_{10}(t), \quad A_2(l_2,t) = A_{20}(t); \\ B_1(0,t) = B_{10}(t), \quad B_2(l_2,t) = B_{20}; \\ H_{1y}(0,t) + \alpha_1 A_{1y}(0,t) = 0; \quad E_{1y}(0,t) + \alpha_1 B_{1y}(0,t) = 0. \end{gathered} \tag{9}$$

The boundary conditions at the interface $y = l_1$ are:

$$\begin{gathered} U_1 = U_2, \quad \rho_2\nu_2 U_{2y} - \rho_1\nu_1 U_{1y} = -2æ_1 A_1 - 2æ_2 H_1, \\ A_1 = A_2, \quad k_1 A_{1y} = k_2 A_{2y}; \\ B_1 = B_2, \quad k_1 B_{1y} = k_2 B_{2y}; \\ H_{1y} + \alpha_1 A_{1y} = 0; \quad E_{1y} + \alpha_1 B_{1y} = 0; \end{gathered} \tag{10}$$

$$\int_0^{l_1} U_1(y,t)dy = 0, \quad \int_{l_1}^{l_2} U_2(y,t)dy = 0. \tag{11}$$

The first from conditions (11) is a consequence of the kinematic conditions, when the interface is stationary, and the second one is the slip condition for velocity components $V_2(y,t)$ on the wall of $y = l_2$.

The initial data are written in the form

$$\begin{gathered} U_j(y,0) = 0, \quad V_j(y,0) = 0, \quad A_j(y,0) = a_j^0(y), \\ B_j(y,0) = b_j^0(y), \quad H_1(y,0) = H^0(y), \quad E_1(y,0) = E^0(y). \end{gathered}$$

Note that this problem is nonlinear and inverse. Because of the function $s_j(t)$ remains unknown as well as functions $U_j, V_j, A_j, B_j, H_1, E_1$.

We introduce the characteristic length scales, time functions $U_j, V_j, P_j, A_j, B_j, H_1, E_1, h_j, s_j$ respectively

$$\begin{gathered} x = l_1\xi, \quad t = \frac{l_1^2}{\nu_1}\tau, \quad U_j = \frac{æ_1\triangle A l_1}{\rho_1\nu_1}U_j^*, \quad V_j = \frac{æ_1\triangle A l_1^2}{\rho_1\nu_1}V_j^*, \quad P_j = æ_1\triangle A l_1 P_j^*, \\ A_j = \triangle A A_j^*, \quad B_j = \triangle A l_1^2 B_j^*, \quad H_1 = \frac{\beta_1^\theta \triangle A}{\beta_1^c}H_1^*, \quad E_1 = \frac{\beta_1^\theta\triangle A l_1^2}{\beta_1^c}E_1^*, \\ h_j = \frac{æ_1\triangle A l_1}{\rho_1}h_j^*, \quad s_j = \frac{æ_1\triangle A}{\rho_1 l_1}s_j^*, \end{gathered} \tag{12}$$

where $\triangle A = \max\limits_{t\geqslant 0} |A_{20}(t) - A_{10}(t)| > 0$. If $A_{20}(t) = A_{10}(t)$ than $\triangle A = \max\limits_j \max\limits_y |A_{j0}(y)| > 0$.

We have the multiplier

$$M = \frac{æ_1\triangle A l_1^3}{\rho_1\nu_1^2} \tag{13}$$

called the Marangoni number at the nonlinear summands in equations (7)–(8) written with dimensionless variables. Let us mention here also Prandtl number Pr_j, Schmidt number Sc_j, parameter $G_j = Gr_j/M$ (where Gr_j is Grashof number), parameter ω and split ratio ψ:

$$Pr_j = \frac{\nu_j}{\chi_j}, \quad Sc_j = \frac{\nu_j}{d_j}, \quad G_j = \frac{g\beta_j^\theta\rho_1 l_1^2}{æ_1}, \quad \omega = \frac{æ_2\beta_1^\theta}{æ_1\beta_1^c}, \quad \psi = -\frac{\alpha\beta_1^c}{\beta_1^\theta}.$$

3. Stationary solution

In the present section we describe stationary solution of (7)–(8). Assume that the motion is creeping in one layers so $M \ll 1$ and parameter $G_j = Gr_j/M = O(1)$. These conditions can be valid in either thin layer or very viscous fluid according to the formula (13).

In such case the steady state problem (7)–(8) has a special form

$$\begin{gathered} \frac{\nu_j}{\nu_1}U_{j\eta\eta} = 2G_j\int_{\Omega_j}(A_j + \frac{\beta_j^c}{\beta_1^c}H_j)d\eta + s_j, \\ A_{j\eta\eta} = 0, \quad B_{j\eta\eta} = -2A_j, \\ H_{1\eta\eta} = 0, \quad E_{1\eta\eta} = -2H_1; \end{gathered} \tag{14}$$

$$
\begin{gathered}
V_{j\eta} = -U_j, \\
\frac{\rho_1}{\rho_j} P_j = \frac{\nu_j}{2\nu_1} U_{j\eta\eta} x^2 + h_j, \\
h_{j\eta} = \frac{\nu_j}{\nu_1} V_{j\eta\eta} + G_j B_j + \frac{\beta_j^c}{\beta_1^c} G_1 E_j.
\end{gathered} \tag{15}
$$

Integrating (14) provides a solution to the problem as the

$$
\begin{gathered}
A_1 = m_1 \eta + m_2, \quad A_2 = m_3 \eta + m_4, \\
B_1 = -\frac{m_1}{3}\eta^3 - m_2 \eta^2 + m_5 \eta + m_6, \quad B_2 = -\frac{m_3}{3}\eta^3 - m_4 \eta^2 + m_7 \eta + m_8, \\
H_1 = m_9 \eta + m_{10}, \quad E_1 = -\frac{m_9}{3}\eta^3 - m_{10}\eta^2 + m_{11}\eta + m_{12}, \\
U_1 = G_1 \left(\frac{m_1 + m_9}{12}\eta^4 + \frac{m_2 + m_{10}}{3}\eta^3 \right) + \frac{s_1}{2}\eta^2 + m_{13}\eta + m_{14}, \\
U_2 = \nu G_2 \left(\frac{m_3}{12}\eta^4 + \frac{m_4}{3}\eta^3 \right) + \frac{\nu s_2}{2}\eta^2 + m_{15}\eta + m_{16}.
\end{gathered} \tag{16}
$$

Constants m_i, $i = \overline{1, 16}$, s_1, s_2 in (16) determined from the boundary conditions (9)–(11) and have the form

$$
m_1 = \frac{l(A_{20} - A_{10})}{l - kl + k}, \quad m_2 = A_{10}, \quad m_3 = k m_1, \quad m_4 = \frac{A_{20}\, l - k m_1}{l}, \quad m_6 = B_{10},
$$

$$
m_5 = \frac{3l^2(2kl - 2k - l)A_{10} - 3l\,(l-1)^2 A_{20} + 3\,l^3 (B_{10} - B_{20}) + m_1\,\left(l\left(kl^2 + 3kl - l^2 - 6\,k\right) + 2k\right)}{3l^2\,(kl - k - l)},
$$

$$
m_7 = \frac{3A_{10}\,kl^3 + 3\,l\left(kl^2 - 2\,l^2 - k\right)A_{20} + 3\,kl^3\,(B_{10} - B_{20}) + k\left(kl^3 - 3\,kl^2 - l^3 + 6\,l^2 + 2\,k\right)m_1}{3l^2\,(kl - k - l)},
$$

$$
\begin{gathered}
m_8 = \frac{3A_{10}kl^2 + 3l\,(kl - k - 2l + 1)\,A_{20} + 3l^2\,(B_{20}l(1-k) - B_{10}k)}{3l^2\,(kl - k - l)} - \\
- \frac{k\left(kl^2 - 3\,kl - l^2 + 2\,k + 6\,l - 2\right)m_1}{3l^2\,(kl - k - l)},
\end{gathered}
$$

$$
m_9 = \psi m_1, \quad m_{10} = \psi A_{10}, \quad m_{11} = \psi m_5, \quad m_{14} = 0,
$$

$$
\begin{gathered}
m_{13} = \frac{(\psi + 1)\,((7\,\rho\,\nu\,(l-1) - 4\,l)\,m_1 + 5\,A_{10}\,(3\,\rho\,\nu\,(l-1) - 2\,l))\,G_1}{60(\rho\,\nu\,l - \rho\,\nu - l)} + \\
+ \frac{\nu\,(l-1)^3\,(3\,k\,(l-1)\,m_1 + 5\,A_{20}\,l)\,G_2}{60l^3\,(\rho\,\nu\,l - \rho\,\nu - l)} - \frac{\nu\,\rho\,(\psi\,\omega + 1)\,(l-1)\,(A_{10} + m_1)}{\rho\,\nu\,l - \rho\,\nu - l},
\end{gathered}
$$

$$
\begin{gathered}
m_{15} = \frac{l\rho\,(\psi + 1)\,(l+2)\,(5\,A_{10} + 3\,m_1)\,\nu\,G_1}{60\,(l\nu\,\rho - \rho\,\nu - l)\,(l-1)} - \frac{l\rho\,(\psi\,\omega + 1)\,(l+2)\,(A_{10} + m_1)\,\nu}{(l\nu\,\rho - \rho\,\nu - l)\,(l-1)} - \\
- \frac{5\,\left(2\,\nu\,(l-1)\left(l^2 + 4\,l + 1\right)\rho - 3\,l^2\,(l+3)\right)A_{20}\nu\,G_2}{60l^2\,(l\nu\,\rho - \rho\,\nu - l)} - \\
- \frac{km_1\,\left(2\,\rho\,(l-1)\left(2\,l^3 + 3\,l^2 - 12\,l - 3\right)\nu - l^2\left(7\,l^2 + 6\,l - 33\right)\right)\nu\,G_2}{60l^3\,(l\nu\,\rho - \rho\,\nu - l)},
\end{gathered}
$$

$$m_{16} = -\frac{\nu\,\rho\,(\psi+1)\,(2\,l+1)\,(5\,A10+3\,m_1)\,G_1 - 60\nu\,\rho\,(\psi\,\omega+1)\,(2\,l+1)\,(A_{10}+m_1)}{120\,(l\nu\,\rho-\rho\,\nu-l)\,(l-1)} +$$
$$+\frac{\nu G_2 A_{20}\,(l-1)\left(20\,l\left(l^2-1\right)\rho\,\nu - 5\,l\left(6\,l^2+3\,l-1\right)\right)}{120 l^3\,(l\nu\,\rho-\rho\,\nu-l)\,(l-1)} +$$
$$+\frac{\nu G_2\,(l-1)\,k\left(2\,(l-1)\left(4\,l^2-3\,l-6\right)\rho\,\nu - 14\,l^3+15\,l^2+12\,l-3\right)m_1}{120 l^3\,(l\nu\,\rho-\rho\,\nu-l)\,(l-1)};$$

$$s_1 = -\frac{(\psi+1)\,((l-1)\,(25\,A_{10}+9\,m_1)\,\rho\,\nu - 2\,l\,(10\,A_{10}+3\,m_1))\,G_1}{20\,(\rho\,\nu\,l-\rho\,\nu-l)} -$$
$$-\frac{\nu\,(l-1)^3\,(3\,k\,(l-1)\,m_1+5\,A_{20}\,l)\,G_2 - 60\,l^3\nu\,\rho\,(\psi\,\omega+1)\,(l-1)\,(A_{10}+m_1)}{20 l^3\,(\rho\,\nu\,l-\rho\,\nu-l)},$$

$$s_2 = -\frac{l^2\rho\,(\psi+1)\,(5\,A_{10}+3\,m_1)\,G_1 - 60\,l^2\rho\,(\psi\,\omega+1)\,(A_{10}+m_1)}{20\,(l-1)\,(l\rho\,\nu-\rho\,\nu-l)} +$$
$$+\frac{(l-1)^2\,(3\,k\,(l-1)\,(2\,\rho\,\nu\,(l-1)-3\,l)\,m_1 + 5\,A_{20}\,l\,(4\,\rho\,\nu\,(l-1)-5\,l))\,G_2}{20\,(l-1)\,(l\rho\,\nu-\rho\,\nu-l)\,l^2}.$$

The expression for V_j, P_j can be obtained with help of formulas (15) and have not shown here because cumbersome.

On Fig. 2a we represent the velocity profiles. In the upper layer the vertical velocity component V for parameters $A_{10} = 0.1$, $A_{20} = -0.3$, $B_{10} = 25$, $B_{20} = 20$, $G_1 = 1.05$, $G_2 = 0.98$, $M = 0.034$ is positive and the horizontal component of the velocity changes sign. The fluid moves vertically when $x = 0$ along the y axis and vertically downward in the lower layer and symmetrically rotated about the axis y (Fig. 2b).

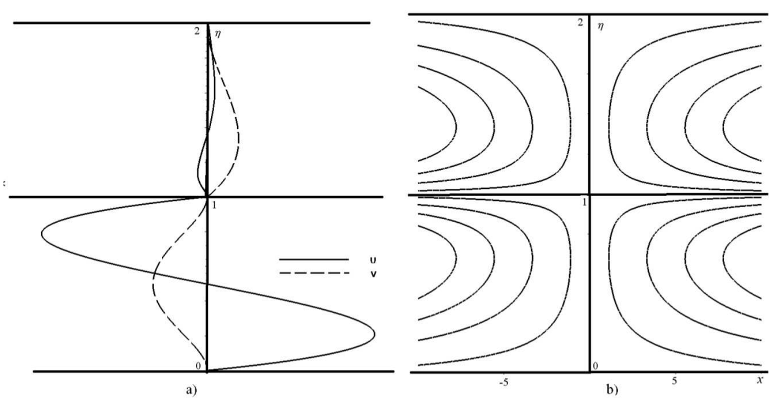

Fig. 2. Profiles of the velocity components U and V (a) and isolines of the stream function $\psi_j = -V_j x$ (b)

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Двумерное стационарное течение бинарной смеси и вязкой жидкости в плоском слое

Марина В. Ефимова

В настоящей статье рассматривается нелинейная модель конвекции в приближении Обербека–Буссинеска, описывающая плоское совместное движение бинарной смеси и вязкой теплопроводной жидкости с общей поверхностью раздела. Важно, что продольный градиент температуры и концентрации имеет квадратичную зависимость от координат x. Построено стационарное решение системы.

Ключевые слова: уравнения Обербека–Буссинеска, конвективное движение, бинарная смесь, установившееся течение.